

Casimir densities in brane models with compact internal spaces

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Abstract

We investigate the Wightman function, the vacuum expectation values of the field squared and the energy-momentum tensor for a massive scalar field with general curvature coupling parameter subject to Robin boundary conditions on two codimension one parallel branes located on $(D+1)$ -dimensional background spacetime $AdS_{D+1} \times \Sigma$ with a warped internal space Σ . The general case of different Robin coefficients on separate branes is considered. Unlike to the purely AdS bulk, the vacuum expectation values induced by a single brane, in addition to the distance from the brane, depends also on the position of the brane in the bulk. The brane induced parts in these expectation values vanish when the brane position tends to the AdS horizon or AdS boundary. For strong gravitational fields corresponding to large values of the AdS energy scale, the both single brane and interference parts of the expectation values integrated over the internal space are exponentially suppressed. An application to the higher dimensional generalization of the Randall-Sundrum brane model with arbitrary mass terms on the branes is discussed. For large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane.

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1 Introduction

The braneworld scenario provides an interesting alternative to the standard Kaluza-Klein compactification of the extra dimensions. The simplest phenomenological models describing such a scenario are the five-dimensional Randall-Sundrum type braneworld models (for a review see [1]). From the point of view of embedding these models into a more fundamental theory, such as string/M-theory, one may expect that a more complete version of the scenario must admit the presence of additional extra dimensions compactified on an internal manifold. From a phenomenological point of view, the consideration of more general spacetimes offer a richer geometrical structure and may provide interesting extensions of the Randall-Sundrum mechanism for the geometric origin of the hierarchy. More extra dimensions also relax the fine-tunings of the fundamental parameters. These models can provide a framework in the context of which the stabilization of the radion field naturally takes place. In addition, a richer topological structure of the field configuration in transverse space provides the possibility of more realistic spectrum of chiral fermions localized on the brane. Several variants of the Randall-Sundrum scenario involving cosmic strings and other global defects of various codimensions have been investigated in higher dimensions (see, for instance, [2] and references therein).

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Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention [3]-[47]. A class of higher dimensional models with the topology $\text{AdS}_{D_1+1} \times \Sigma$, where Σ is a one-parameter compact manifold, and with two branes of codimension one located at the orbifold fixed points, is considered in Refs. [25, 26]. In both cases of the warped and unwarped internal manifold, the quantum effective potential induced by bulk scalar fields is evaluated and it has been shown that this potential can stabilize the hierarchy between the Planck and electroweak scales without fine tuning. In addition to the effective potential, the investigation of local physical characteristics in these models is of considerable interest. Local quantities contain more information on the vacuum fluctuations than the global ones and play an important role in modelling a self-consistent dynamics involving the gravitational field. In papers [39, 40, 41] we have studied the bulk and surface Casimir densities for a scalar field with an arbitrary curvature coupling parameter obeying Robin boundary conditions on two codimension one parallel branes embedded in the background spacetime $\text{AdS}_{D_1+1} \times \Sigma$ with a warped internal space Σ . For an arbitrary internal space Σ , the application of the generalized Abel-Plana formula [48] allowed us to extract from the vacuum expectation values the part due to the bulk without branes and to present the brane induced parts in terms of exponentially convergent integrals for the points away from the branes. In the present paper we review these results.

The paper is organized as follows. In the next section we evaluate the Wightman function in the region between the branes. By using the generalized Abel-Plana formula, we present this function in the form of a sum of the Wightman function for the bulk without boundaries and boundary induced parts. The vacuum expectation value of the bulk energy-momentum tensor for a general case of the internal space Σ is discussed in section 3. The interaction forces between the branes are discussed in section 4. The surface Casimir densities and the energy balance are considered in section 5. The last section contains a summary of the work.

2 Wightman function

For a free scalar field $\varphi(x)$ with curvature coupling parameter ζ the equation of motion has the form

$$(g^{MN} \nabla_M \nabla_N + m^2 + \zeta R) \varphi(x) = 0, \quad (1)$$

where $M, N = 0, 1, \dots, D$, and R is the scalar curvature. We will assume that the background spacetime has a topology $\text{AdS}_{D_1+1} \times \Sigma$, where Σ is a D_2 -dimensional compact manifold. The corresponding line element has the form

$$ds^2 = g_{MN} dx^M dx^N = e^{-2k_D y} \eta_{\mu\nu} dx^\mu dx^\nu - e^{-2k_D y} \gamma_{ij} dX^i dX^j - dy^2, \quad (2)$$

with $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$ being the metric for the D_1 -dimensional Minkowski spacetime $R^{(D_1-1,1)}$ and the coordinates X^i cover the manifold Σ , $D = D_1 + D_2$. Here and below $\mu, \nu = 0, 1, \dots, D_1 - 1$ and $i, j = 1, \dots, D_2$. The scalar curvature for the metric tensor from (2) is given by the expression $R = -D(D+1)k_D^2 - e^{2k_D y} R_{(\gamma)}$, where $R_{(\gamma)}$ is the scalar curvature for the metric tensor γ_{ik} . In the discussion below, in addition to the coordinate y we will use the radial coordinate z defined by the relation $z = e^{k_D y}/k_D$. In terms of the coordinate z , the metric tensor is conformally related to the metric of the direct product space $R^{(D_1,1)} \times \Sigma$ by the conformal factor $(k_D z)^{-2}$.

Our main interest in this paper will be the Wightman function and the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor induced by two infinite parallel branes of codimension one with the coordinates $y = a$ and $y = b$, $a < b$. We will assume that on this branes the scalar field obeys the boundary conditions

$$(\tilde{A}_j + \tilde{B}_j \partial_y) \varphi(x) = 0, \quad y = j, \quad j = a, b, \quad (3)$$

with constant coefficients \tilde{A}_j, \tilde{B}_j . In the orbifolded version of the model which corresponds to a higher dimensional Randall-Sundrum braneworld these coefficients are expressed in terms of the surface mass parameters and the curvature coupling of the scalar field. In quantum field theory the imposition of boundary conditions modifies the spectrum for the zero-point fluctuations and as a result the VEVs for physical observables are changed. These effects can either stabilize or destabilize the braneworlds and have to be taken into account in the self-consistent formulation of the braneworld dynamics.

As a first stage in the investigations of local quantum effects, we will consider the positive frequency Wightman function defined as the expectation value $G^+(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle$. In the region between the branes, $a < y < b$, the Wightman function is presented as the mode-sum:

$$G^+(x, x') = \frac{k_D^{D-1} (zz')^{D/2}}{2^{D_1+1} \pi^{D_1-3} z_a^2} \sum_{\beta} \psi_{\beta}(X) \psi_{\beta}^*(X') \int d\mathbf{k} e^{i\mathbf{k}\Delta\mathbf{x}} \times \sum_{n=1}^{\infty} \frac{h_{\beta\nu}(u)}{[A_b^2 + B_b^2(\eta^2 u^2 - \nu^2)] \bar{J}_{\nu}^{(a)2}(u) / \bar{J}_{\nu}^{(b)2}(\eta u) - A_a^2 + B_a^2(u^2 - \nu^2)} \Big|_{u=\gamma_{\nu,n}}, \quad (4)$$

where $\mathbf{x} = (x^1, x^2, \dots, x^{D_1-1})$ represents the spatial coordinates in $R^{(D_1-1,1)}$, $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$, $\eta = z_b/z_a$, and

$$h_{\beta\nu}(u) = u g_{\nu}(u, uz/z_a) g_{\nu}(u, uz'/z_a) \frac{e^{-i\Delta t \sqrt{u^2/z_a^2 + k^2 + \lambda_{\beta}^2}}}{\sqrt{u^2/z_a^2 + k^2 + \lambda_{\beta}^2}}, \quad (5)$$

$$g_{\nu}(u, v) = J_{\nu}(v) \bar{Y}_{\nu}^{(a)}(u) - \bar{J}_{\nu}^{(a)}(u) Y_{\nu}(v), \quad \nu = \sqrt{(D/2)^2 - D(D+1)\zeta + m^2/k_D^2}, \quad (6)$$

with $k = |\mathbf{k}|$, $\Delta t = t - t'$, $z_j = e^{k_D j}/k_D$, $j = a, b$, $J_{\nu}(x)$, $Y_{\nu}(x)$ are the Bessel and Neumann functions. In formula (6), for a given function $F(x)$ we use the notation

$$\bar{F}^{(j)}(x) = A_j F(x) + B_j x F'(x), \quad A_j = \tilde{A}_j + \tilde{B}_j k_D D/2, \quad B_j = \tilde{B}_j k_D, \quad j = a, b. \quad (7)$$

In the discussion below we will assume values of the curvature coupling parameter for which ν is real. For imaginary ν the ground state becomes unstable [49]. In (4), the modes $\psi_{\beta}(X)$ are the eigenfunctions for the operator $\Delta_{(\gamma)} + \zeta R_{(\gamma)}$:

$$[\Delta_{(\gamma)} + \zeta R_{(\gamma)}] \psi_{\beta}(X) = -\lambda_{\beta}^2 \psi_{\beta}(X), \quad \int d^{D_2} X \sqrt{\gamma} \psi_{\beta}(X) \psi_{\beta'}^*(X) = \delta_{\beta\beta'}, \quad (8)$$

with eigenvalues λ_{β}^2 , and $\Delta_{(\gamma)}$ is the Laplace-Beltrami operator for the metric γ_{ij} . From the boundary condition on the branes we receive that the eigenvalues $\gamma_{\nu,n}$ have to be solutions to the equation

$$g_{\nu}^{(ab)}(\gamma_{\nu,n}, \eta \gamma_{\nu,n}) \equiv \bar{J}_{\nu}^{(a)}(\gamma_{\nu,n}) \bar{Y}_{\nu}^{(b)}(\eta \gamma_{\nu,n}) - \bar{Y}_{\nu}^{(a)}(\gamma_{\nu,n}) \bar{J}_{\nu}^{(b)}(\eta \gamma_{\nu,n}) = 0. \quad (9)$$

This equation determines the tower of radial Kaluza-Klein (KK) masses.

Applying to the sum over n in (4) a variant of the generalized Abel-Plana formula [48], the Wightman function is presented in two equivalent forms ($j = a, b$)

$$G^+(x, x') = G_0^+(x, x') + \langle \varphi(x) \varphi(x') \rangle^{(j)} - \frac{k_D^{D-1} (zz')^{D/2}}{2^{D_1-1} \pi^{D_1}} \sum_{\beta} \psi_{\beta}(X) \psi_{\beta}^*(X') \times \int d\mathbf{k} e^{i\mathbf{k}\Delta\mathbf{x}} \int_{\sqrt{k^2 + \lambda_{\beta}^2}}^{\infty} du u G_{\nu}^{(j)}(uz_a, uz) G_{\nu}^{(j)}(uz_a, uz') \times \frac{\Omega_{j\nu}(uz_a, uz_b)}{\sqrt{u^2 - k^2 - \lambda_{\beta}^2}} \cosh(\Delta t \sqrt{u^2 - k^2 - \lambda_{\beta}^2}), \quad (10)$$

where $I_\nu(u)$ and $K_\nu(u)$ are the modified Bessel functions and

$$\begin{aligned}\Omega_{av}(u, v) &= \frac{\bar{K}_\nu^{(b)}(v)/\bar{K}_\nu^{(a)}(u)}{\bar{K}_\nu^{(a)}(u)\bar{I}_\nu^{(b)}(v) - \bar{K}_\nu^{(b)}(v)\bar{I}_\nu^{(a)}(u)}, \\ \Omega_{bv}(u, v) &= \frac{\bar{I}_\nu^{(a)}(u)/\bar{I}_\nu^{(b)}(v)}{\bar{K}_\nu^{(a)}(u)\bar{I}_\nu^{(b)}(v) - \bar{K}_\nu^{(b)}(v)\bar{I}_\nu^{(a)}(u)}, \\ G_\nu^{(j)}(u, v) &= I_\nu(v)\bar{K}_\nu^{(j)}(u) - \bar{I}_\nu^{(j)}(u)K_\nu(v), \quad j = a, b.\end{aligned}\tag{11}$$

In (10), the term

$$\begin{aligned}G_0^+(x, x') &= \frac{k_D^{D-1}(zz')^{\frac{D}{2}}}{2^{D_1}\pi^{D_1-1}} \sum_\beta \psi_\beta(X)\psi_\beta^*(X') \int d\mathbf{k} e^{i\mathbf{k}\Delta\mathbf{x}} \\ &\times \int_0^\infty du u \frac{e^{-i\Delta t \sqrt{u^2 + k^2 + \lambda_\beta^2}}}{\sqrt{u^2 + k^2 + \lambda_\beta^2}} J_\nu(uz) J_\nu(uz'),\end{aligned}\tag{12}$$

does not depend on the boundary conditions and is the Wightman function for the $AdS_{D_1+1} \times \Sigma$ spacetime without branes. The second term on the right of Eq. (10) is given by the formula

$$\begin{aligned}\langle \varphi(x)\varphi(x') \rangle^{(a)} &= -\frac{k_D^{D-1}(zz')^{\frac{D}{2}}}{2^{D_1-1}\pi^{D_1}} \sum_\beta \psi_\beta(X)\psi_\beta^*(X') \int d\mathbf{k} e^{i\mathbf{k}\Delta\mathbf{x}} \\ &\times \int_{\sqrt{k^2 + \lambda_\beta^2}}^\infty du u \frac{\bar{I}_\nu^{(a)}(uz_a)}{\bar{K}_\nu^{(a)}(uz_a)} \frac{K_\nu(uz)K_\nu(uz')}{\sqrt{u^2 - k^2 - \lambda_\beta^2}} \cosh(\Delta t \sqrt{u^2 - k^2 - \lambda_\beta^2}),\end{aligned}\tag{13}$$

for $j = a$, and the expression for $\langle \varphi(x)\varphi(x') \rangle^{(b)}$ is obtained from (13) by the replacements $a \rightarrow b$, $I_\nu \rightleftharpoons K_\nu$. The term $\langle \varphi(x)\varphi(x') \rangle^{(j)}$ does not depend on the parameters of the brane at $z = z_{j'}$, $j' \neq j$, and is induced by a single brane at $z = z_j$ when the boundary $z = z_{j'}$ is absent. In the same way described above for the Wightman function, any other two-point function can be evaluated. Note that the expression for the Wightman function is not symmetric with respect to the interchange of the brane indices. The reason for this is that the boundaries have nonzero extrinsic curvature tensors and two sides of the boundaries are not equivalent. In particular, for the geometry of a single brane the VEVs are different for the regions on the left and on the right of the brane. In the region $y < a$ the Wightman has the form $G^+(x, x') = G_0^+(x, x') + \langle \varphi(x)\varphi(x') \rangle^{(a)}$, where the expression for the second term on the right hand-side is obtained from (13) by the replacement $I_\nu \rightleftharpoons K_\nu$. Similarly, for the Wightman function in the region $y > b$ one has $G^+(x, x') = G_0^+(x, x') + \langle \varphi(x)\varphi(x') \rangle^{(b)}$, where the second term is given by formula (13) replacing $a \rightarrow b$.

In the higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk $AdS_{D_1+1} \times \Sigma$ the Wightman function for untwisted scalar is given by formula (10) with an additional factor $1/2$ and with Robin coefficients

$$\tilde{A}_a/\tilde{B}_a = -c_a/2 - 2D\zeta k_D, \quad \tilde{A}_b/\tilde{B}_b = c_b/2 - 2D\zeta k_D.\tag{14}$$

For twisted scalar field Dirichlet boundary conditions are obtained. The one-loop effective potential and the problem of moduli stabilization in this model with zero mass parameters c_j are discussed in Ref. [25].

3 Vacuum energy-momentum tensor

The VEV of the energy-momentum tensor can be evaluated by substituting the Wightman function and the VEV of the field squared into the formula

$$\langle 0|T_{MN}|0\rangle = \lim_{x' \rightarrow x} \partial_M \partial'_N G^+(x, x') + \left[\left(\zeta - \frac{1}{4} \right) g_{MN} \nabla_L \nabla^L - \zeta \nabla_M \nabla_N - \zeta R_{MN} \right] \langle 0|\varphi^2|0\rangle, \quad (15)$$

where R_{MN} is the Ricci tensor. Substituting the expression for the Wightman function into this formula, for the components of the vacuum energy-momentum tensor in the region between the branes we obtain the formula

$$\begin{aligned} \langle 0|T_M^N|0\rangle &= \langle T_M^N \rangle^{(0)} + \langle T_M^N \rangle^{(j)} - \frac{2k_D^{D+1} z^D}{(4\pi)^{D_1/2} \Gamma(D_1/2)} \sum_{\beta} |\psi_{\beta}(X)|^2 \\ &\times \int_{\lambda_{\beta}}^{\infty} du u (u^2 - \lambda_{\beta}^2)^{\frac{D_1}{2}-1} \Omega_{j\nu}(uz_a, uz_b) F_{\beta M}^{(+N)} [G_{\nu}^{(j)}(uz_j, uz)], \end{aligned} \quad (16)$$

with the functions $F_{\beta M}^{(+N)}[g(v)]$, $g(v) = G_{\nu}^{(j)}(uz_j, v)$, defined by the relations

$$F_{\beta\mu}^{(\pm)\sigma}[g(v)] = \delta_{\mu}^{\sigma} \left(\frac{1}{4} - \zeta \right) \left\{ z^2 g^2(v) \eta_{\beta}(X) + 2v \frac{\partial}{\partial v} F[g(v)] + \frac{\pm v^2 - z^2 \lambda_{\beta}^2}{D_1(\zeta - 1/4)} g^2(v) \right\}, \quad (17)$$

$$\begin{aligned} F_{\beta D}^{(\pm)D}[g(v)] &= \left(\frac{1}{4} - \zeta \right) z^2 g^2(v) \eta_{\beta}(X) + \frac{1}{2} [-v^2 g'^2(v) \\ &+ D(4\zeta - 1)vg(v)g'(v) + (2m^2/k_D^2 - \nu^2 \pm v^2) g^2(v)], \end{aligned} \quad (18)$$

for the components in the AdS part, and by the relations

$$F_{\beta D}^{(\pm)i}[g(v)] = \frac{k_D}{2} z^2 (1 - 4\zeta) F[g(v)] \eta_{\beta}^i(X), \quad (19)$$

$$F_{\beta i}^{(\pm)k}[g(v)] = z^2 g^2(v) \frac{t_{\beta i}^k(X)}{|\psi_{\beta}(X)|^2} + \frac{1}{2} \delta_i^k (1 - 4\zeta) v \frac{\partial}{\partial v} F[g(v)], \quad (20)$$

with $t_{\beta i}^k(X) = -\gamma^{kl} t_{\beta il}(X)$, for the components having indices in the internal space. In these expressions we use the following notations

$$F[g(v)] = vg(v)g'(v) + \frac{1}{2} \left(D + \frac{4\zeta}{4\zeta - 1} \right) g^2(v), \quad (21)$$

$$\eta_{\beta}(X) = \frac{\Delta_{(\gamma)} |\psi_{\beta}(X)|^2}{|\psi_{\beta}(X)|^2}, \quad \eta_{\beta}^i(X) = -\gamma^{ik} \frac{\partial_k |\psi_{\beta}(X)|^2}{|\psi_{\beta}(X)|^2}, \quad (22)$$

$$\begin{aligned} t_{\beta ik}(X) &= \nabla_{(\gamma)i} \psi_{\beta}(X) \nabla_{(\gamma)k} \psi_{\beta}^*(X) + \\ &\left[\left(\zeta - \frac{1}{4} \right) \gamma_{ik} \Delta_{(\gamma)} - \zeta \nabla_{(\gamma)i} \nabla_{(\gamma)k} - \zeta R_{(\gamma)ik} \right] |\psi_{\beta}(X)|^2, \end{aligned} \quad (23)$$

where $\nabla_{(\gamma)i}$ is the covariant derivative operator associated with the metric tensor γ_{ik} .

In formula (16),

$$\langle T_M^N \rangle^{(0)} = \frac{k_D^{D+1} z^D}{(4\pi)^{\frac{D_1}{2}}} \Gamma \left(1 - \frac{D_1}{2} \right) \sum_{\beta} |\psi_{\beta}(X)|^2 \int_0^{\infty} du u (u^2 + \lambda_{\beta}^2)^{\frac{D_1}{2}-1} F_{\beta M}^{(-N)} [J_{\nu}(uz)], \quad (24)$$

is the VEV for the energy-momentum tensor in the background without branes, and the term $\langle T_M^N \rangle^{(j)}$ is induced by a single brane at $z = z_j$. For the left brane one has

$$\langle T_M^N \rangle^{(a)} = -\frac{2k_D^{D+1} z^D}{(4\pi)^{\frac{D_1}{2}} \Gamma(\frac{D_1}{2})} \sum_{\beta} |\psi_{\beta}(X)|^2 \int_{\lambda_{\beta}}^{\infty} du u(u^2 - \lambda_{\beta}^2)^{\frac{D_1}{2}-1} \frac{\bar{I}_{\nu}^{(a)}(uz_a)}{\bar{K}_{\nu}^{(a)}(uz_a)} F_{\beta M}^{(+N)}[K_{\nu}(uz)], \quad (25)$$

and the corresponding expression for the right brane is obtained by the replacements $a \rightarrow b$, $I_{\nu} \rightleftharpoons K_{\nu}$. Unlike to the case of purely AdS bulk, here the VEVs for a single brane in addition to the distance from the brane depend also on the position of the brane in the bulk. In the limit when the AdS curvature radius tends to infinity we derive the formula for the vacuum energy-momentum tensor for parallel plates on the background spacetime with topology $R^{(D_1,1)} \times \Sigma$. In this limit for a homogeneous internal space $\frac{D}{D}$ -component of the brane induced part in the VEV of the energy-momentum tensor vanishes.

The features of the single brane parts in the VEVs in the asymptotic regions of the parameters are as follows. For the points on the brane the vacuum energy-momentum tensor diverges. Near the brane the total vacuum energy-momentum tensor is dominated by the brane induced part and has opposite signs for Dirichlet and non-Dirichlet boundary conditions. Near the brane $\frac{D}{D}$ - and $\frac{i}{D}$ -components of this tensor have opposite signs in the regions $y < a$ and $y > a$. For large distances from the brane in the region $y > a$ the contribution of a given mode along Σ with nonzero KK mass is suppressed by the factor $e^{-2\lambda_{\beta} z}$. For the zero mode the brane induced VEV near the AdS horizon behaves as $z^{D_2-2\nu}$. In the purely AdS bulk ($D_2 = 0$) this VEV vanishes on the horizon for $\nu > 0$. For an internal spaces with $D_2 > 2\nu$ the VEV diverges on the horizon. The VEV integrated over the internal space vanishes on the AdS horizon for all values D_2 due to the additional warp factor coming from the volume element. For the points near the AdS boundary, the brane induced VEV vanishes as $z^{D+2\nu}$ for diagonal components and as $z^{D+2\nu+2}$ for the $\frac{i}{D}$ -component. For small values of the length scale for the internal space, the contribution of nonzero KK masses is exponentially suppressed and the main contribution into the brane induced energy-momentum tensor comes from the zero mode. In the opposite limit, when the length scale of the internal space is large, to the leading order the vacuum energy-momentum tensor reduces to the corresponding result for a brane in the bulk AdS_{D+1} given in Ref. [36]. For strong gravitational fields corresponding to small values of the AdS curvature radius, the contribution from nonzero KK modes along Σ is suppressed by the factor $e^{-2\lambda_{\beta}|z-z_a|}$. For the zero KK mode the components of the brane induced vacuum energy-momentum tensor behave like $k_D^{D_1+1} e^{D_2 k_D y} \exp[(D_1 + 2\nu)k_D(y - a)]$ in the region $y < a$ and like $k_D^{D_1+1} e^{D_2 k_D y} \exp[2\nu k_D(a - y)]$ in the region $y > a$. The corresponding quantities integrated over the internal space contain additional factor $e^{-D_2 k_D y}$ coming from the volume element and are exponentially small in both regions. For fixed values of the other parameters, the brane induced VEV in the region $y > a$ vanishes as $z_a^{2\nu}$ when the brane position tends to the AdS boundary. When the brane position tends to the AdS horizon, $z_a \rightarrow \infty$, for massive KK modes along Σ the VEV of the energy-momentum tensor in the region $z < z_a$ is suppressed by the factor $e^{-2z_a \lambda_{\beta}}$. For the zero mode in the same limit the suppression is power-law with respect to z_a .

For the geometry of two branes, the VEV in the region between the branes is presented as

$$\langle 0|T_M^N|0 \rangle = \langle T_M^N \rangle^{(0)} + \sum_{j=a,b} \langle T_M^N \rangle^{(j)} + \langle T_M^N \rangle^{(ab)}, \quad (26)$$

with separated boundary-free, single branes and interference parts. The latter is finite everywhere including the points on the branes. The surface divergences are contained in the single brane parts only. The both single brane and interference parts separately satisfy the continuity equation and are traceless for a conformally coupled massless scalar. The possible trace anomalies are contained in the boundary-free parts. In the limit $k_D \rightarrow 0$ we derive the corresponding results for two parallel Robin

plates in the bulk $R^{(D_1,1)} \times \Sigma$. For small values of the length scale of the internal space corresponding to large KK masses, the interference part in the VEV of the energy-momentum tensor is suppressed by the factor $e^{-2\lambda_\beta(z_b - z_a)}$. The interference part vanishes as $z_a^{2\nu}$ when the left brane tends to the AdS boundary. Under the condition $z \ll z_b$ an additional suppression factor appears in the form $(z/z_b)^{D_1}$ for D -component and in the form $(z/z_b)^{D_1+2\alpha_1}$ for the other components, where $\alpha_1 = \min(1, \nu)$.

4 Interaction forces

Now we turn to the investigations of the vacuum forces acting on the branes. The corresponding effective pressure $p^{(j)}$ acting on the brane at $z = z_j$ is determined by D -component of the vacuum energy-momentum tensor evaluated at the point of the brane location: $p^{(j)} = -\langle T_D^D \rangle_{z=z_j}$. For the region between two branes it can be presented as a sum of two terms: $p^{(j)} = p_1^{(j)} + p_{(\text{int})}^{(j)}$, $j = a, b$. The first term is the pressure for a single brane at $z = z_j$ when the second brane is absent. This term is divergent due to the surface divergences in the VEVs and needs additional renormalization. This can be done, for example, by applying the generalized zeta function technique to the corresponding mode-sum. Below we will be concentrated on the term $p_{(\text{int})}^{(j)}$. This term is the additional vacuum pressure induced by the presence of the second brane, and can be termed as an interaction force. It is determined by the last term on the right of formulae (16) evaluated at the brane location $z = z_j$. It is finite for all nonzero interbrane distances and is not changed by the renormalization procedure. Substituting $z = z_j$ into the second term on the right of formula (16), for the interaction part of the vacuum effective pressure one finds

$$p_{(\text{int})}^{(j)} = \frac{k_D^{D+1} z_j^D}{(4\pi)^{\frac{D_1}{2}} \Gamma(\frac{D_1}{2})} \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^\infty du u (u^2 - \lambda_\beta^2)^{\frac{D_1}{2}-1} \Omega_{j\nu}(uz_a, uz_b) F_\beta^{(j)}(uz_j), \quad (27)$$

where we have introduced the notation

$$F_\beta^{(j)}(u) = (u^2 - \nu^2 + 2m^2/k_D^2) B_j^2 - D(4\zeta - 1) A_j B_j - A_j^2 - 2(\zeta - 1/4) z_j^2 B_j^2 \eta_\beta(X). \quad (28)$$

For small interbrane distances the interaction part dominates the single brane parts. For a Dirichlet scalar $\Omega_{j\nu}(uz_a, uz_b) > 0$ and the vacuum interaction forces are attractive. For a given value of the AdS energy scale k_D and one parameter manifold Σ with size L , the vacuum interaction forces (27) are functions on the ratios z_b/z_a and L/z_a . The first ratio is related to the proper distance between the branes and the second one is the ratio of the size of the internal space measured by an observer residing on the brane at $y = a$ to the AdS curvature radius k_D^{-1} . The quantity $p_{(\text{int})}^{(j)}$ determines the force by which the scalar vacuum acts on the brane due to the modification of the spectrum for the zero-point fluctuations by the presence of the second brane. As the vacuum properties depend on the coordinate y , there is no a priori reason for the interaction terms to be equal for the branes $j = a$ and $j = b$, and the corresponding forces in general are different even in the case of the same Robin coefficients in the boundary conditions.

Taking the limit $k_D \rightarrow 0$ we obtain the result for the interaction forces between two Robin plates in the bulk $R^{(D_1-1,1)} \times \Sigma$. In this case, for a homogeneous internal space the interaction forces are the same even in the case of different Robin coefficients for separate branes. For the modes along Σ with large KK masses, the interaction forces are exponentially small. In particular, for sufficiently small length scales of the internal space this is the case for all nonzero KK modes and the main contribution to the interaction forces comes from the zero mode. For small interbrane distances, the interaction forces are repulsive for Dirichlet boundary condition on one brane and non-Dirichlet boundary condition on the another and are attractive for other cases. For small interbrane distances the contribution of the interaction term dominates the single brane parts, and the same is the case

for the total vacuum forces acting on the branes. When the right brane tends to the AdS horizon, $z_b \rightarrow \infty$, the interaction force acting on the left brane vanishes as $e^{-2\lambda_\beta z_b}/z_b^{D_1/2}$ for the nonzero KK mode and like $z_b^{-D_1-2\nu}$ for the zero mode. In the same limit the corresponding force acting on the right brane behaves as $z_b^{D_2+D_1/2+1}e^{-2\lambda_\beta z_b}$ for the nonzero KK mode and like $z_b^{D_2-2\nu}$ for the zero mode. In the limit when the left brane tends to the AdS boundary the contribution of a given KK mode into the vacuum interaction force vanishes as $z_a^{D+2\nu}$ and as $z_a^{2\nu}$ for the left and right branes, respectively. For small values of the AdS curvature radius corresponding to strong gravitational fields, under the conditions $\lambda_\beta z_a \gg 1$ and $\lambda_\beta(z_b - z_a) \gg 1$, the contribution to the interaction forces is suppressed by the factor $e^{-2\lambda_\beta(z_b - z_a)}$. For the zero KK mode, the corresponding interaction forces integrated over the internal space behave as $k_D^{D_1+1} \exp[(D_1\delta_j^a + 2\nu)k_D(a - b)]$ for the brane at $y = j$ and are exponentially small. In the model without the internal space the suppression is relatively weaker.

5 Surface energy-momentum tensor

On manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary. For an arbitrary smooth boundary ∂M with the inward-pointing unit normal vector n^L , the surface part of the energy-momentum tensor for a scalar field is given by the formula [50] $T_{MN}^{(s)} = \delta(x; \partial M) \tau_{MN}$, where the 'one-sided' delta-function $\delta(x; \partial M)$ locates this tensor on ∂M and

$$\tau_{MN} = \zeta \varphi^2 K_{MN} - (2\zeta - 1/2) h_{MN} \varphi n^L \nabla_L \varphi. \quad (29)$$

In this formula, h_{MN} is the induced metric on the boundary and K_{MN} is the corresponding extrinsic curvature tensor. From the point of view of physics on the brane at $y = j$, Eq. (29) corresponds to the gravitational source of the cosmological constant type with the surface energy density $\varepsilon_j^{(s)} = \langle 0 | \tau_0^{(j)0} | 0 \rangle$ (surface energy per unit physical volume on the brane at $y = j$ or brane tension), stress $p_j^{(s)} = -\langle 0 | \tau_1^{(j)1} | 0 \rangle$, and the equation of state $\varepsilon_j^{(s)} = -p_j^{(s)}$. It is noteworthy that this relation takes place for both subspaces on the brane.

For two-brane geometry the VEV of the surface energy density on the brane at $y = j$ is presented as the sum $\varepsilon_j^{(s)} = \varepsilon_{1j}^{(s)} + \Delta \varepsilon_j^{(s)}$. The first term on the right is the energy density induced on a single brane when the second brane is absent. This part is evaluated in [41] by using the generalized zeta function method. The second term is induced by the presence of the second brane and is given by the formula

$$\Delta \varepsilon_j^{(s)} = \frac{2C_j n^{(j)} (k_D z_j)^D B_j^2}{(4\pi)^{D_1/2} \Gamma(D_1/2)} \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^\infty du u (u^2 - \lambda_\beta^2)^{\frac{D_1}{2}-1} \Omega_{j\nu}(u z_a, u z_b), \quad (30)$$

with the notation $C_j = \zeta - (2\zeta - 1/2) \tilde{A}_j / (k_D \tilde{B}_j)$. As we consider the region $a \leq y \leq b$, the energy density $\varepsilon_j^{(s)}$ is located on the surface $y = a + 0$ for the left brane and on the surface $y = b - 0$ for the right brane. The energy densities on the surfaces $y = a - 0$ and $y = b + 0$ are the same as for the corresponding single brane geometry. For an observer living on the brane at $y = j$ the corresponding effective D_1 -dimensional cosmological constant is determined by the relation

$$\Lambda_{D_1j} = 8\pi M_{D_1j}^{2-D_1} e^{-D_2 k_D j} \int_\Sigma d^{D_2} X \sqrt{\gamma} \Delta \varepsilon_j^{(s)}, \quad (31)$$

where M_{D_1j} is the D_1 -dimensional effective Planck mass scale for the same observer. In Ref. [41] it has been shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. In the Randall-Sundrum braneworld model, for the interbrane distances solving the hierarchy problem between the gravitational

and electroweak mass scales, the cosmological constant generated on the visible brane is of the right order of magnitude with the value suggested by the cosmological observations.

On background of manifolds with boundaries the total vacuum energy is splitted into bulk and boundary parts. In the region between two branes the bulk energy per unit coordinate volume in the D_1 -dimensional subspace is obtained by the integration of the 0 -component of the volume energy-momentum tensor over this region: $E^{(v)} = \int d^{D_2} X dy \sqrt{|g|} \langle 0 | T_0^{(v)0} | 0 \rangle$. The surface energy per unit coordinate volume in the D_1 -dimensional subspace, $E^{(s)}$, is related to the surface densities by the formula $E^{(s)} = \sum_{j=a,b} (k_D z_j)^{-D} \varepsilon_j^{(s)}$. Now it can be seen that the formal relation $E = E^{(v)} + E^{(s)}$ takes place for the unrenormalized VEVs, where

$$E = \frac{1}{2} \int \frac{d^{D_1-1} \mathbf{k}}{(2\pi)^{D_1-1}} \sum_{\beta} \sum_{n=1}^{\infty} (k^2 + m_n^2 + \lambda_{\beta}^2)^{1/2}, \quad m_n = \gamma_{\nu,n}/z_a, \quad (32)$$

is the total vacuum energy per unit coordinate volume of the D_1 -dimensional subspace, evaluated as the sum of zero-point energies of elementary oscillators. The latter can be presented in the form $E = \sum_{j=a,b} E_j + \Delta E$, where E_a (E_b) is the vacuum energy for the geometry of a single brane at $y = a$ ($y = b$) in the region $y \geq a$ ($y \leq b$), and the interference term is given by the formula

$$\Delta E = \sum_{\beta} \int_{\lambda_{\beta}}^{\infty} du \frac{u(u^2 - \lambda_{\beta}^2)^{D_1/2-1}}{(4\pi)^{D_1/2} \Gamma(D_1/2)} \ln \left| 1 - \frac{\bar{I}_{\nu}^{(a)}(uz_a) \bar{K}_{\nu}^{(b)}(uz_b)}{\bar{K}_{\nu}^{(a)}(uz_a) \bar{I}_{\nu}^{(b)}(uz_b)} \right|. \quad (33)$$

The total vacuum energy within the framework of the Randall-Sundrum braneworld is evaluated in Refs. [8, 6, 15] by the dimensional regularization method and in Ref. [11] by the zeta function technique. Refs. [8, 6, 11] consider the case of a minimally coupled scalar field in $D = 4$, and the case of arbitrary ζ and D with zero mass terms c_a and c_b is discussed in Ref. [15]. For the orbifolded version of the model under consideration with $D_1 = 4$ and zero mass terms on the branes, the vacuum energy is investigated in [25] by using the dimensional regularization. The zeta function approach in the general case is considered in [41].

Now let us check that for the separate parts of the vacuum energy the standard energy balance equation takes places. We denote by P the perpendicular vacuum stress on the brane integrated over the internal space. This stress is determined by the vacuum expectation value of the $\frac{D}{D}$ -component of the bulk energy-momentum tensor: $P = - \int d^{D_2} X \sqrt{\gamma} \langle 0 | T_D^{(v)D} | 0 \rangle$. In the presence of the surface energy the energy balance equation is in the form

$$dE = -PdV + \sum_{j=a,b} E_j^{(s)} dS^{(j)}, \quad E_j^{(s)} = \int d^{D_2} X \sqrt{\gamma} \varepsilon_j^{(s)}, \quad (34)$$

where V is the $(D+1)$ -volume in the bulk and $S^{(j)}$ is the D -volume on the brane $y = j$ per unit coordinate volume in the D_1 -dimensional subspace:

$$V = \int_a^b dy e^{-Dk_D y} \int d^{D_2} X \sqrt{\gamma}, \quad S^{(j)} = e^{-Dk_D j} \int d^{D_2} X \sqrt{\gamma}, \quad j = a, b. \quad (35)$$

It can be explicitly checked that the separate parts in the vacuum energies and effective pressures on the branes obey the equation (34).

6 Conclusion

From the point of view of embedding the braneworld model into a more fundamental theory one may expect that a more complete version of this scenario must admit the presence of additional extra

dimensions compactified on a manifold Σ . In the present paper we have considered the local vacuum effects in the braneworlds with the AdS bulk on a higher dimensional brane models which combine both the compact and warped geometries. This problem is also of separate interest as an example with gravitational, topological, and boundary polarizations of the vacuum, where one-loop calculations can be performed in closed form. We have investigated the Wightman function and the bulk and surface Casimir densities for a scalar field with an arbitrary curvature coupling parameter satisfying Robin boundary conditions on two parallel branes in $AdS_{D_1+1} \times \Sigma$ spacetime. In the region between the branes the KK modes corresponding to the radial direction are zeros of a combination of the cylinder functions. The application of the generalized Abel-Plana formula to the corresponding mode sum allowed us to extract from the VEVs the boundary-free part and to present the brane induced parts in terms of integrals rapidly convergent in the coincidence limit of the arguments. We give an application of our results to the higher dimensional version of the Randall-Sundrum braneworld with arbitrary mass terms on the branes. For the untwisted scalar the Robin coefficients are expressed through these mass terms and the curvature coupling parameter by formulae (14). For the twisted scalar Dirichlet boundary conditions are obtained on both branes.

In the model under discussion the hierarchy between the fundamental Planck scale and the effective Planck scale in the brane universe is generated by the combination of redshift and large volume effects. For large interbrane separations the corresponding effective Newton's constant on the brane at $y = b$ is exponentially small. This mechanism also allows obtaining a naturally small cosmological constant generated by the vacuum quantum fluctuations of a bulk scalar. In [41] we have considered two classes of models with the compactification scale on the visible brane close to the fundamental Planck scale. For the first one the higher dimensional Planck mass and the AdS inverse radius are of the same order and in the second one a separation between these scales is assumed. In both cases the corresponding interbrane distances generating the hierarchy between the electroweak and Planck scales are smaller than those for the model without an internal space and the required suppression of the cosmological constant is obtained without fine tuning.

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